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Wrinkle-free solutions in the theory of curved circular membranes

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In honour of the late Professor H.J. Weinitschke, Ph.D.

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Abstract. Rotationally symmetric deformations of a curved circular elastic membrane under a vertical surface load are studied, with prescribed radial stresses or radial displacements at the edge. Considering Reissner's theory of thin shells of revolution suffering small strains but arbitrarily large deflections and rotations, the determination of the principal stresses in the membrane is shown to be equivalent to the solution of a single, second-order ODE, expressed in terms of a geodesic variable. Analytical techniques are applied in order to determine a limit curve of those boundary data, which subdivide the parameter range into complementary domains of existence and non-existence of tensile solutions. Finally, the more restricted subdomain of those boundary parameters is determined which admit wrinkle-free solutions, i.e. solutions governed by a nonnegative radial and circumferential stress component.

1. Introduction

In this paper we discuss rotationally symmetric deformations of a curved circular membrane subject to a vertical surface load (e.g. gravity). At the membrane edge either radial stresses or radial displacements are prescribed. We ask for that boundary data which is related to a stable equilibrium of the membrane. According to D. Steigmann [1] a necessary and sufficient condition for stability is that both principal stresses should be nonnegative everywhere in the membrane. A stable state prevents the membrane from buckling. Since buckling of a stretched membrane is termed 'wrinkling', the solutions corresponding to a stable equilibrium will be called *wrinkle-free*.

The present analysis of wrinkle-free solutions will be developed within the framework of E. Reissner's theory of thin shells of revolution having negligible bending stiffness [2]. Assuming small strains, the linear stress-strain relations are determined by Hooke's law, while strain-displacement relations are genuinely nonlinear because arbitrarily large but finite displacements and rotations are admitted. So the Reissner theory is geometrically nonlinear.

The problem of curved membranes under a variable vertical load has apparently not been investigated as yet. The results of the present paper originate from A. Beck's doctoral thesis [3], however with an important improvement of the representation by introducing a geodesic variable and with an extension of some uniqueness results.

Special configurations of curved membranes have been discussed by J.V. Baxley [4] and R.W. Dickey [5, 6] under the more restricted assumption of small finite deflections. This approach which is known as the Föppl-Hencky theory [7,8] becomes invalid for large deformations and thus results are not directly comparable to results of the Reissner theory, except for a very restricted parameter range.

The method employed in our work consists of using an integral equation technique for an existence proof for solutions of the stress boundary value problem and a maximum principle for a proof of positivity and uniqueness. Integral equation methods have first been applied to nonlinear membrane problems by Dickey [9] and later successfully been refined by adding concavity and monotonicity arguments. Equipped with these tools, the mathematical problems of existence and uniqueness of a stable membrane state were solved for flat circular and annular membranes not only within the small finite deflection theory [10, 11, 12] but also within a simplified version of the Reissner theory of finite rotations [13, 14, 15, 16, 17]. The results are summarized in [18].

Concerning the displacement boundary value problem, we employ a mapping argument which originates from R. Pirner's diploma thesis [19] and which has been improved later on by H. Grabmüller et al. [14, 15, 16]. The idea of this method consists of mapping the set of parameters related to a stable state of the stress problem into the parameter set associated with the displacement problem.

Since the flat circular membrane is incorporated in our analysis as a special case $p \equiv 1$ (see Section 2) and since our considerations are not restricted to the simplified Reissner model, the results of the present paper constitute thereby a significant extension of previous work on flat circular membranes.

It should be noted that some work on deformation of curved membranes with finite rotations under uniform normal pressure was done by J. Arango [20]. In his thesis, a quite different approach is pursued, namely a shooting method which has previously been introduced in membrane theory by A.J. Callegary and E.L. Reiss [21].

The paper is organized as follows. In Section 2 the basic equations of the nonlinear membrane theory are supplied, following the representation of R.A. Clark and O.S. Narayanaswamy [22]. A geodesic variable is used which simplifies the analysis considerably. After having introduced the notation of regular tensile solutions the stress problem is solved in Section 3. In Section 4 the concept of subsolutions is employed for a proof of some monotonicity results which are needed in the mapping argument. In Section 5 the displacement problem is discussed and in a final section we examine the actual parameter set related to the wrinkle-free solutions of both the stress and displacement problem.

2. The basic equations

Using cylindrical coordinates (r, θ, ζ), the undeformed membrane constitutes a surface of revolution in three-dimensional space, parametrized in the form $r = r(s)$, $\zeta = \zeta(s)$. The arclength along a meridian of the surface is denoted by $s, 0 \leqslant s \leqslant L < \infty$. Following the notation of R.A. Clark and O.S. Narayanaswamy [22], the state of the deformed membrane is completely described by the quantities

$$M = rS_\Phi \cos \Phi\ , \quad N = rS_\Phi \sin \Phi\ , \quad H = rS_\theta\ , \tag{2.1}$$

where S_Φ and S_θ are pseudo stress resultants measuring the tension in the deformed membrane per unit length of the undeformed membrane. The angle Φ of inclination is measured from the radial axis to the meridional tangent of the deformed membrane.

The deformed membrane is stably equilibrated in the sense of D. Steigmann [1], if both stresses S_Φ and S_θ are nonnegative everywhere. Since rotations are restricted to $0 \leqslant |\Phi(s)| \leqslant \frac{\pi}{2}$, $0 \leqslant s < L$, the radial and circumferential stresses M and H have to be nonnegative for wrinkle-free solutions. We assume that a vertical surface load $q_\zeta(s)$ is distributed per unit undeformed area. This leads to the following system of nonlinear ODEs for the unknowns M, N and H (see [22] for details)

$$
\begin{aligned}
& r\frac{\mathrm{d}}{\mathrm{d}s}M = H\,, \\
& r\frac{\mathrm{d}}{\mathrm{d}s}N = -r^2 q_\zeta\,, \qquad\qquad (2.2)\\
& r\frac{\mathrm{d}}{\mathrm{d}s}H = M + Ehr\Big(\frac{M}{\sqrt{M^2+N^2}} - \frac{\mathrm{d}}{\mathrm{d}s}r\Big) - \nu r^2 q_\zeta \frac{N}{\sqrt{M^2+N^2}}\,.
\end{aligned}
$$

Here $E > 0$ denotes Young's modulus of elasticity, ν Poisson's ratio (restricted to $0 \leqslant \nu \leqslant 1/2$) and $h > 0$ the thickness of the membrane.

The undeformed membrane should have smoothness $r \in C^2[0, L]$. Since the membrane is closed at the apex we necessarily have $r(0) = 0$ and $r'(0) = 1$, while simple geometrical considerations lead to the requirement that $0 \leqslant r(s) \leqslant s$, $s \in [0, L]$, and $1 \geqslant r'(s) > 0$, $s \in [0, L)$. From this it is justified to assume that

$$r(s) = \frac{s}{1 + s r_1(s)}\,, \quad s \in [0, L]\,, \quad 0 \leqslant r_1 \in C^2[0, L]\,, \qquad (2.3)$$

$$r'(s) = 1 - s r_2(s)\,, \quad s \in [0, L]\,, \quad 0 \leqslant r_2(s) < \frac{1}{s}\,, \quad s \in (0, L)\,, \quad r_2 \in C^1[0, L]\,. \qquad (2.4)$$

Using the geodesic variable

$$t := \exp\Big(-\int_s^L \frac{\mathrm{d}\tau}{r(\tau)}\Big) = \frac{s}{L}\exp\Big(-\int_s^L r_1(\tau)\,\mathrm{d}\tau\Big)\,, \qquad s \in [0, L]\,, \qquad (2.5)$$

it becomes obvious that t establishes a C^2-diffeomorphism from the segment $[0, L]$ onto $[0, 1]$ with $\mathrm{d}t/\mathrm{d}s = t/r > 0$. Furthermore, letting $C := Eh$, $tp(t) := r(s)/L$, $q(t) := Lq_\zeta(s)$ and $(x(t), y(t), z(t)) := (M(s), N(s), H(s))/L$, the system (2.2) can be written for $0 < t < 1$ in the dimensionless form

$$
\begin{aligned}
& tx'(t) = z(t)\,, \\
& ty'(t) = -t^2p^2(t)q(t)\,, \qquad\qquad (2.6)\\
& tz'(t) = x(t) + Ctp(t)\Big(\frac{x(t)}{\sqrt{x^2(t)+y^2(t)}} - 1 - \frac{tp'(t)}{p(t)}\Big) - \frac{\nu t^2p^2(t)q(t)y(t)}{\sqrt{x^2(t)+y^2(t)}}\,,
\end{aligned}
$$

where primes are used to denote derivatives with respect to t. If the ν-term in (2.6c) is neglected, we get the *simplified Reissner model* (cf. [23], [24]). Numerical computations show that this term does not have a significant influence on solutions, if C is sufficiently large.

Since $r(0)=0$, it follows from (2.1) that $x(0)=y(0)=z(0)=0$ holds. Thus, equation (2.6b) yields by direct integration:

$$y(t) = -\int_0^t sp^2(s)q(s)\,\mathrm{d}s\,, \quad t \in [0,1]\,. \tag{2.7}$$

Notice, that the function $p(t)$ has smoothness $p \in C^2[0,1]$ and that

$$p'(t) = -\frac{Lr_2(s)}{1+sr_1(s)}\exp\Big(2\int_s^L r_1(s)\,\mathrm{d}s\Big) \leq 0\,, \quad t \in [0,1]\,. \tag{2.8}$$

Therefore, p is bounded by $0 < p(1) \leq p(t) \leq p(0)$.

The following assumption (Q) prevents $y(t)$ from vanishing at an inner point $t \in (0,1)$ and should be valid throughout the whole paper:

The load function $q \in C[0,1]$ *satisfies* $q(t) \geq 0$, $t \in [0,1]$, *and* $q_0 := q(0) > 0$. (Q)

Since $y(t)$ is a known function, an elimination of $z(t)$ from the system (2.6) yields a single second order nonlinear ODE of the form

$$[Lx](t) := t^2x''(t) + tx'(t) = x(t) + F(t, x(t))\,, \quad t \in (0,1)\,, \tag{2.9}$$

where

$$F(t,x) := Ctp(t)\Big(\frac{x}{\sqrt{x^2+y^2(t)}} - 1 - \frac{tp'(t)}{p(t)}\Big) - \nu t^2p^2(t)q(t)\frac{y(t)}{\sqrt{x^2+y^2(t)}}\,. \tag{2.10}$$

The radial displacement of a membrane point after deformation is given by [22]

$$\tilde{u} = \frac{1}{Eh}\Big(r\frac{\mathrm{d}}{\mathrm{d}s}M - \nu\sqrt{M^2+N^2}\Big)\,.$$

Letting $u(t) := (C/L)\tilde{u}(s)$, we obtain $u(t) = tx'(t) - \nu\sqrt{x^2(t)+y^2(t)}$. Now, at the membrane edge $t=1$ either the radial stress $x(1)=S$ or the radial displacement $u(1)=H$ may be prescribed. These conditions together with $x(0)=0$ and the differential equation (2.9) constitute a set of two nonlinear boundary value problems (BVPs), and reference shall be made to them as the BVP (S) or (H). In this paper, only nonnegative solutions $x(t)$ are of interest (the so-called *tensile* solutions). Thus, in the sequel the concept of determining the *regular tensile solutions* (rt-solutions for short, i.e. functions $x \in C^2(0,1) \cap C^1[0,1]$ satisfying $x(t) > 0$ for $t \in (0,1]$) will be pursued. A function $x(t)$ with the smoothness of an rt-solution but with a less strict positivity $x(t) > 0$, $t \in (0,1)$, will be called a *regular nonnegative solution* (rn-solution).

3. Tensile solutions of the BVP (S)

In the sequel, I_0 denotes the segment $0 < t < 1$ and I its closure, while E denotes the Banach space $C(I)$ of real-valued continuous functions over I, equipped with the usual max-norm $\|\cdot\|$.

Using Green's function, it follows by standard arguments that the solutions $x \in C^2(I_0) \cap C(I)$ of the BVP (S) coincide with the solutions $x \in E$ of the integral equation

$$x(t) = St - \int_0^1 K(t,s) f(s, x(s))\, ds\,, \quad t \in I\,, \tag{3.1}$$

where $f(t,x) := F(t,x)/t$ and where the kernel function

$$K(t,s) := \begin{cases} \frac{1}{2}(1-t^2)\frac{s}{t}\,, & 0 \leq s \leq t \leq 1\,, \\ \frac{1}{2}(1-s^2)\frac{t}{s}\,, & 0 \leq t \leq s \leq 1\,, \end{cases} \tag{3.2}$$

is nonnegative and bounded above by 1/2. In order to supply a result on existence of a continuous solution of (3.1) it is convenient to introduce for fixed $S \in \mathbb{R}$ the nonlinear operator $T{:}E \to E$ by

$$[Tx](t) := St - \int_0^1 K(t,s) f(s, x(s))\, ds\,, \quad x \in E\,, \quad t \in I\,, \tag{3.3}$$

and to consider the subset $M_S \subset E$ defined by

$$M_S := \{x \in E \mid \exists H \in C(I_0){:}\ |H(t)| \leq K_1 \quad \text{and} \quad x(t) := St - \int_0^1 K(t,s) H(s)\, ds\}\,,$$

where $K_1 := \|C(2p - tp') + \nu tp^2 q\| < \infty$. It can easily be seen that any solution $x \in E$ of (3.1), if it exists, belongs to M_S. Moreover, the set M_S is convex, and thus the Schauder fixed point theorem (see e.g. [25], §7, XII) may be applied to T to establish a first existence result for solutions of equation (3.1).

LEMMA 3.1. *For each fixed $S \in \mathbb{R}$ there exists at least one solution $x \in M_S$ of the integral equation* (3.1).

Proof: As we are going to show, the operator T is a continuous mapping from E to E which obviously satisfies $T(\overline{M_S}) \subseteq M_S$. In order to prove the continuity, define a regularization $T_n, n \in \mathbb{N}$, by

$$[T_n x](t) := St - \int_0^1 K(t,s) f_n(s, x(s))\, ds\,, \quad x \in E\,, \quad t \in I\,,$$

where $f_n(t,x)$ is function $f(t,x)$ with $\sqrt{x^2 + y^2(t)}$ replaced by $\sqrt{x^2 + y^2(t) + n^{-2}}$. We observe that each $T_n{:}E \to E$ is continuous and that

$$\lim_{n\to\infty} \sup_{\substack{\|x\| \leq 1 \\ x \in E}} \|Tx - T_n x\| \leq \frac{1}{2} \lim_{n\to\infty} \int_0^1 \frac{Cp(s) + \nu sp^2(s) q(s)}{\sqrt{n^2 y^2(s) + 1}}\, ds = 0$$

holds. Thus, the convergence $T_n \to T$ is uniform which proves the continuity of T. In order to make Schauder's fixed theorem applicable to T, we finally observe that the set $T(\overline{M_s})$ is relatively compact according to the theorem of Arzela-Ascoli (see e.g. [25], §7, IV). Therefore, T has at least one fixed point $x \in M_S$. □

Solutions $x \in E$ of equation (3.1) are by no means rn-solutions of the BVP (S), not even for $S \geq 0$. In general, $x(t)$ suffers from missing positivity as well as the additional C^1-smoothness at the endpoints of the interval I. In order to secure positivity, we require a somewhat sharper condition for $q(t)$, namely:

In addition to (Q) *the load function satisfies*

$$q(t) < \frac{C(tp(t))'}{\nu t p^2(t)} = \frac{CL}{\nu} \frac{\mathrm{d}}{\mathrm{d}s} \ln r(s) , \quad t \in I_0 . \tag{Q1}$$

Condition (Q1) is void, if the simplified Reissner model is considered. On the other hand, numerical computations show that solutions $x(t)$ of the BVP (S) fail to be positive, if $S \geq 0$ is small and if condition (Q1) is violated. However, we have

LEMMA 3.2. *Suppose condition* (Q1) *holds. Then, for each fixed* $S \geq 0$, *there exists at least one solution* $x \in C^2(I_0) \cap C(I)$, $x(t) > 0$, $t \in I_0$, *of the* BVP (S).

Proof: Take $x(t)$ to be any solution of equation (3.1) that exists by Lemma 3.1. Then $x(t)$ is a solution of the BVP (S) with the required regularity inherent. Suppose there exists a local minimum $x(t_0) \leq 0$ at an inner point $t_0 \in I_0$. Then $x'(t_0) = 0$ and $x''(t_0) \geq 0$, in contradiction to

$$x''(t_0) \leq \nu p^2(t_0) q(t_0) - \frac{C}{t_0} (t_0 p(t_0))' < 0 , \tag{3.4}$$

which follows from equation (2.9) and from condition (Q1). □

The nonnegative solutions of equation (3.1) have already the required C^1-smoothness of rn-solutions. A proof of this statement will be prepared by the following

LEMMA 3.3. *Suppose condition* (Q1) *holds. For fixed* $S \geq 0$, *let* $x \in E$ *denote a solution of equation* (3.1). *Then*:

(a) *The integral* $I(t) := \int_t^1 (1 - s^2) \frac{1}{s} f(s, x(s)) \, \mathrm{d}s$ *exists for all* $t \in I$.

(b) *There exists a constant* K_2 *such that* $0 \leq x(t)/t \leq K_2 < \infty$, $t \in I$, *holds.*

Proof: (a) For $t \in (0, 1)$, it follows from (3.1) and from Lemma 3.2 that

$$0 \leq \frac{x(t)}{t} = S - \frac{1}{2t^2} (1 - t^2) \int_0^t s f(s, x(s)) \, \mathrm{d}s - \frac{1}{2} I(t) . \tag{3.5}$$

According to $|f(t, x(t))| \leq K_1$, $t \in I$, the first integral on the right hand side of (3.5) is bounded by $K_1/4$. Concerning the existence of $I(t)$, we make use of the decomposition

$$I(t) = C \int_t^1 \Big(\frac{x(s)}{\sqrt{x^2(s) + y^2(s)}} - 1 \Big) \frac{p(s) \, \mathrm{d}s}{s} - \int_t^1 \Big(C p'(s) + \frac{\nu p^2(s) q(s) y(s)}{\sqrt{x^2(s) + y^2(s)}} + F(s, x(s)) \Big) \, \mathrm{d}s .$$

The second integral is continuous at each point $t \in I$ because it integrates a continuous function. The first integral is continuous and monotone increasing on $(0, 1]$. Suppose we have $\lim_{t\to 0+} I(t) = -\infty$. By (3.5), $x(t)/t$ becomes unbounded above as $t \to 0+$. Thus, there exists a number $\varepsilon > 0$ with

$$\left|\frac{y(t)}{x(t)}\right| = \left|\frac{y(t)}{t}\right| \Big/ \frac{x(t)}{t} \leq t \quad \text{for all} \quad t \in (0, \varepsilon) .$$

From this we obtain, as a contradiction, the existence of the integral

$$0 \geq \int_0^\varepsilon \left(\frac{x(s)}{\sqrt{x^2(s) + y^2(s)}} - 1\right) \frac{p(s)\,\mathrm{d}s}{s} \geq p(0) \int_0^\varepsilon \left(\frac{1}{\sqrt{1+s^2}} - 1\right) \frac{\mathrm{d}s}{s} > -\infty .$$

Thus, $\lim_{t\to 0+} I(t) = I(0) > -\infty$ exists, and the assertion (a) is proved.
(b) This assertion follows from (a) with $K_2 := S + K_1/4 + \|I(t)\|/2$. □

We are now in a position to establish the first main result on existence of rn-solutions to the BVP (S).

THEOREM 3.4. *Suppose condition* (Q1) *holds. Then, for each fixed* $S \geq 0$, *there exists at least one rn-solution* $x(t)$ *of the* BVP (S). *For* $S > 0$, $x(t)$ *is an rt-solution.*

Proof: We need only to prove the C^1-smoothness of $x(t)$ at the edge points $t = 0$ and $t = 1$. This, however, is easily done at $t = 1$, because differentiation of (3.1) yields for each $t \in (0, 1]$:

$$x'(t) = S - \int_0^1 \partial_t K(t, s) f(s, x(s))\,\mathrm{d}s . \tag{3.6}$$

Using this and the relation (3.5), it follows from L'Hospital's rule and from Lemma 3.3(a) that

$$\begin{aligned}\lim_{t\to 0+} x'(t) &= S - \frac{1}{2} I(0) = S - \lim_{t\to 0+} \frac{1}{2t^2}\left((1-t^2)\int_0^t sf(s, x(s))\,\mathrm{d}s + t^2 I(t)\right)\\ &= \lim_{t\to 0+} \frac{x(t)}{t} = x'(0) .\end{aligned}$$

So the regularity $x \in C^2(I_0) \cap C^1(I)$ is proved. □

The discussion of uniqueness of rn-solutions is unexpectedly difficult and we did not succeed in solving this problem in general. But a partial result can be derived from an application of the generalized maximum principle of Protter and Weinberger [26]. For this purpose, we consider $F(t, x)$ for fixed $t \in I_0$ as a function of $x \geq 0$. Of main interest is the sign of the quantity

$$D(x) := \frac{\mathrm{d}}{\mathrm{d}x} F(t, x) = tp(t) \frac{Cy^2(t) + \nu tp(t)q(t)y(t)x}{(x^2 + y^2(t))^{3/2}} . \tag{3.7}$$

In case of the simplified Reissner model we have $D(x) \geq 0$. In general, however, $D(x)$ may become negative for some values of $x > 0$. Indeed, since $D(0) \geq 0$ and since $D(x) \to 0$ as $x \to +\infty$, a global negative minimum is attained at

$$x_0 = \frac{|y|}{4}(3\lambda^{-1} + \sqrt{9\lambda^{-2} + 8}),$$

where we have used the notation $\lambda = \lambda(t) := \nu t p(t) q(t)/C > 0$. This follows from calculating the zeros of $D'(x)$. If the continuous and monotone decreasing function $g(\lambda)$ is introduced by

$$g(\lambda) := \frac{\sqrt{6}}{9} \frac{\sqrt{9 + 8\lambda^2} - 3}{\lambda^2(3 + 4\lambda^2 + \sqrt{9 + 8\lambda^2})^{1/2}},$$

and if we let $Y(t) := \int_0^t p(s)\lambda(s)\,ds$, it follows that

$$D(x_0) = -\frac{\nu t p(t)\lambda^3(t)}{Y(t)} g(\lambda(t)) \leq 0. \tag{3.8}$$

Due to the monotonicity of $g(\lambda)$, we have $0 \leq g(\lambda) \leq g(0) = 4/27$, while $g(\lambda) \to 0$ as $\lambda \to +\infty$. Equipped with these prerequisites, we are prepared to establish a sufficient condition for the uniqueness of rn-solutions, namely:

There exist two numbers α, β with $\alpha = 0$, $\alpha = 1$ or $\alpha \geq 2$ and $0 \leq \beta \leq 1$ such that for all $t \in I_0$

$$\alpha^2 t^\alpha + \beta - \frac{\nu t p(t)\lambda^3(t)}{Y(t)} g(\lambda(t)) \geq 0. \tag{$\mathrm{Q}_{\alpha,\beta}$}$$

Now, the result on uniqueness can be formulated as follows.

THEOREM 3.5. *Suppose condition* $(\mathrm{Q}_{\alpha,\beta})$ *holds and let $S \geq 0$ be fixed. Then there exists at most one rn-solution of the* BVP (S).

Proof: Using e.g. power series expansions in terms of t^α, it is immediately verified that the second order ODE $Lw + \alpha^2 t^\alpha w = 0$ possesses a regular solution $w \in C^2(I)$ with $w(t) > 0$, $t \in I$. For $\alpha = 0$, this is the constant function $w = 1$. Supposed, there are two rn-solutions x_1 and x_2, we let $d(t) := x_1(t) - x_2(t), t \in I$. A pointwise application of the mean value theorem leads to

$$[Ld](t) = d(t) + F(t, x_1(t)) - F(t, x_2(t)) = (1 + D(x_\mu(t)))\, d(t), \quad t \in I_0,$$

where the intermediate variable $x_\mu \geq 0$ is defined by $x_\mu(t) := x_1(t) + \mu(t)(x_2(t) - x_1(t))$, $0 < \mu(t) < 1$. Observe, that $d(0) = 0 = d(1)$. We define the new dependent variable $v := d/w$ and we let $h(t) := \alpha^2 t^\alpha + 1 + D(x_\mu(t))$. A simple computation yields

$$t^2 w(t) v'' + (2t^2 w'(t) + t w(t)) v' = h(t) w(t) v. \tag{3.9}$$

From $(\mathrm{Q}_{\alpha,\beta})$ it follows that $h(t) \geq 0$, while $v(0) = 0 = v(1)$. Thus, equation (3.9) can be subjected to the maximum principle (see [26], Thm. 3), from which we obtain $v \equiv 0$ and therefore, $x_1 = x_2$. □

REMARKS: (a) It should be noted that the preceding theorem does not make any use of the assertion (Q1). Furthermore, up to this point the condition $q(0) > 0$ has not influenced

any parts of the existence and uniqueness results and may be cancelled from (Q). However, it becomes important in later considerations on monotonicity.
(b) Condition $(\mathrm{Q}_{\alpha,\beta})$ may be replaced by the stronger condition

$$\alpha^2 t^\alpha + \beta - \frac{4}{27}\frac{\nu t p(t)\lambda^3(t)}{Y(t)} \geqslant 0\,, \quad t \in I_0\,, \qquad (\mathrm{Q}'_{\alpha,\beta})$$

which is independent of $g(\lambda)$ and thus easier to check.
(c) If, in addition, (Q1) is satisfied, then it follows that $\lambda(t) \leqslant 1 + tp'(t)/p(t) \leqslant 1$. While $\lambda^2 g(\lambda)$ is monotone increasing for $\lambda \geqslant 0$, condition $(\mathrm{Q}_{\alpha,\beta})$ may also be replaced by

$$\alpha^2 t^\alpha + \beta - \frac{\nu t p(t)\lambda(t)}{Y(t)}\, g(1) \geqslant 0\,, \quad t \in I_0\,, \qquad (\mathrm{Q}''_{\alpha,\beta})$$

where $g(1) \doteq 0.0917$.
(d) The important case of a uniform load $q(t) \equiv q_0 > 0$ is covered by condition $(\mathrm{Q}''_{\alpha,\beta})$, $\alpha := 0$, $\beta := g(1)$, since

$$\beta Y(t) - \nu t p(t)\lambda(t) g(1) = \frac{\nu q_0 g(1)}{C}\int_0^t sp(s)(p(s) - 2\nu(sp(s))')\,\mathrm{d}s \geqslant 0\,.$$

Likewise, $(\mathrm{Q}''_{\alpha,\beta})$ holds for each monotone decreasing load $q(t)$ satisfying (Q1).
(e) Given a load of the form $q(t) = q_0 t^\gamma$, $\gamma > 0$, we obtain

$$Y(t) = \frac{\nu q_0}{C}\int_0^t s^{1+\gamma} p^2(s)\,\mathrm{d}s \geqslant \frac{\nu q_0 p(t)}{C}\frac{p(t)t^{2+\gamma}}{2+\gamma}\,.$$

If condition (Q1) holds, it follows that $\nu q_0 p(t)/C \leqslant 1$. In this case, condition $(\mathrm{Q}'_{\alpha,\beta})$ is valid with $\alpha := 2(1+\gamma)$ and $\beta := 0$.

4. Properties of rn-solutions of the BVP (S)

The main purpose of this section is to show that rn-solutions $x(t)$ of the BVP (S) monotonely depend on the boundary parameter $S \geqslant 0$. As a first step in that direction we introduce the concept of *subsolutions* of the BVP (S). Here, a function $u \in C^2(I_0) \cap C^1(I)$ is called a subsolution, if $0 < u(t) \leqslant x(t)$, $t \in I_0$, and $u(0) = 0$, $u(1) = S$ is valid for any rn-solution $x(t)$ of the BVP (S). Concerning the existence of subsolutions, we have the following

LEMMA 4.1. *Suppose condition* (Q1) *holds and let* $S \geqslant 0$ *be fixed. Define* $P(t) := -Ctp'(t) + \nu t p^2(t) q(t) \geqslant 0$. *Then there exists a unique solution* $u(t)$ *of the boundary value problem*

$$Lu - u = G(t,u) := Ctp(t)\Big(\frac{u}{\sqrt{u^2 + y^2(t)}} - 1\Big) + tP(t)\,, \quad t \in I_0\,, \qquad (\underline{\mathrm{S}})$$

$$u(0) = 0\,, \quad u(1) = S\,,$$

which is a subsolution of the BVP (S).

Proof: As in Section 3, an equivalent form of the BVP $(\underline{\mathrm{S}})$ is the integral equation

$$u(t) = St - \int_0^1 K(t,s)g(s,u(s))\,\mathrm{d}s\,, \quad t \in I\,, \tag{4.1}$$

where $g(t,u) := G(t,u)/t$. The existence of a solution $u(t)$ with the required smoothness and the positivity $u(t) > 0$, $t \in I_0$, follows analogously like in Lemma 3.1, Lemma 3.2 and Theorem 3.4. The maximum principle (cf. proof of Thm. 3.5) delivers uniqueness of $u(t)$. Choosing any rn-solution $x(t)$ of the BVP (S), the function $d(t) := x(t) - u(t)$ satisfies the relations $d(0) = 0 = d(1)$ and the second order ODE

$$[Ld](t) = d(t)\Big(1 + \frac{Ctp(t)y^2(t)}{(x_\theta^2(t) + y^2(t))^{3/2}}\Big) - \nu t^2 p^2(t)q(t)\Big(1 - \frac{|y(t)|}{\sqrt{x^2(t) + y^2(t)}}\Big)\,,$$

where $x_\theta(t) := u(t) + \theta(t)(x(t) - u(t))$, $0 < \theta(t) < 1$. An application of the maximum principle like in Lemma 3.2 yields $\mathrm{d}(t) > 0$, $t \in I_0$, and therefore $u(t)$ is a subsolution. □

Each rn-solution $x(t)$ of the BVP (S) starts at the edge $t = 0$ with a *positive* slope. In a proof of this important feature the existence of subsolutions is used.

LEMMA 4.2. *Suppose condition* (Q1) *holds and let* $x(t)$ *denote an rn-solution of the* BVP (S). *Then* $x'(0) > 0$.

Proof: Since the subsolution $u(t)$ of the BVP ($\underline{\mathrm{S}}$) evidently satisfies $x'(0) \geqslant u'(0) \geqslant 0$, it suffices to show that the assumption $u'(0) = 0$ is contradictory. The function $P(t)$ defined in Lemma 4.1 satisfies for some finite constant K_3 the relation $0 \leqslant P(t)/t \leqslant K_3$, $t \in I$, and therefore an elementary computation shows that

$$H(t) := \frac{t}{2}\int_0^t \Big(1 - \frac{s^2}{t^2}\Big)\frac{P(s)\,\mathrm{d}s}{s} \geqslant 0\,, \quad t \in I\,,$$

with the regularity $H \in C^2(I_0) \cap C^1(I)$ is the unique solution of the initial value problem $Lx - x = tP(t)$, $t \in I_0$, $x(0) = x'(0) = 0$. Observing that $|y(t)| > 0$ for $0 < t \leqslant 1$, the function $v(t) := u(t) - H(t)$ solves the differential inequality

$$v''(t) + \frac{1}{t}v'(t) - \frac{1}{t^2}v(t) = \frac{Cp(t)}{t}\Big(\frac{u(t)}{\sqrt{u^2(t) + y^2(t)}} - 1\Big) < 0\,, \quad t \in I_0\,, \tag{4.2}$$

as well as the initial conditions $v(0) = v'(0) = 0$. Notice that $H(t)/t^2 \leqslant K_3/3$. Thus,

$$v(t) \geqslant t^2\Big(\frac{u(t)}{t^2} - \frac{K_3}{3}\Big)\,, \quad t \in I\,,$$

which will be used to show that $v(\varepsilon) > 0$ holds for a number $0 < \varepsilon \leqslant 1$. At first, for the same reasons as in Lemma 3.3, the integral

$$0 \geqslant \int_t^1 \Big(\frac{u(s)}{\sqrt{u^2(s) + y^2(s)}} - 1\Big)\frac{p(s)\,\mathrm{d}s}{s} = \int_t^1 \left(\frac{\frac{u(s)}{s^2}}{\sqrt{\Big(\frac{u(s)}{s^2}\Big)^2 + \Big(\frac{y(s)}{s^2}\Big)^2}} - 1\right)\frac{p(s)\,\mathrm{d}s}{s}$$

must exist at each point $t \in I$. Next, because of condition (Q), we have $y(t)/t^2 \to -p^2(0)q_0/2$

<0 as $t \to 0+$. Therefore, $u(t)/t^2$ has to grow unboundedly as $t \to 0+$ and hence $u(\varepsilon)/\varepsilon^2 > K_3/3$ holds for some $0 < \varepsilon \leq 1$. Now, assume that $v(t)$ has a local minimum $v(t_0) \leq 0$ at $t_0 \in (0, \varepsilon)$. From (4.2) we obtain $v''(t_0) < 0$ which contradicts the minimum conditions. Thus, $v(t)$ must be strictly positive on the segment $(0, \varepsilon]$, attaining a minimum $v(0) = 0$ at the boundary point $t = 0$. Since the coefficients of the differential inequality (4.2) satisfy all requirements of Theorem 4 in [26], an application of this theorem to $-v$ yields $v'(0) > 0$. So we have reached a contradiction to the assumption $v'(0) = u'(0) = 0$. □

The next lemma is concerned with the boundedness of the partial derivatives $f_x(t, x)$ and $f_{xx}(t, x)$, where $f(t, x)$ is defined as in Section 3.

LEMMA 4.3. *Suppose condition* (Q1) *holds and let $x(t)$ denote any rn-solution of the* BVP (S). *Then there exist constants* $0 < K_4, K_5 < \infty$, *depending on $x(t)$, such that*

$$|f_x(t, x(t))| \leq K_4\,, \quad |f_{xx}(t, x(t))| \leq K_5\,, \quad t \in I\,.$$

In particular, we have $f_x(t, x(t)) = O(t)$ *as* $t \to 0+$.

Proof: Using $f_x(t, x) = t^{-1}(\mathrm{d}/\mathrm{d}x)F(t, x)$, it becomes obvious from (3.7) that $|f_x(t, x(t))|$ is a continuous and thus bounded function on every subset $[\varepsilon, 1] \subset I$, $\varepsilon > 0$. Furthermore, since $x(t)/t \to x'(0) > 0$ as $t \to 0+$, it follows that

$$f_x(t, x(t)) = tp(t)\frac{C\left(\frac{y(t)}{t^2}\right)^2 + \nu p(t)q(t)\frac{y(t)}{t^2}\frac{x(t)}{t}}{\left(\left(\frac{x(t)}{t}\right)^2 + \left(\frac{y(t)}{t}\right)^2\right)^{3/2}} \to 0 \quad \text{as} \quad t \to 0+\,. \tag{4.3}$$

Hence, continuity extends to $t = 0$, while the $O(t)$-relation follows from $f_x(t, x(t))/t \to (p^4(0)q_0^2/4x'^3(0))(Cp(0) - 2\nu x'(0))$ as $t \to 0+$. In order to prove the boundedness of $|f_{xx}|$, we calculate

$$f_{xx}(t, x) = p(t)\frac{\nu tp(t)q(t)y(t)(y^2(t) - 2x^2) - 3Cy^2(t)x}{(x^2 + y^2(t))^{5/2}}\,.$$

Again it becomes obvious that $|f_{xx}(t, x(t))|$ is continuous except for $t = 0$. However, continuity extends to $t = 0$ by virtue of

$$\lim_{t \to 0+} f_{xx}(t, x(t)) = -\frac{p^4(0)q_0^2}{x'^4(0)}\left(\frac{3}{4}Cp(0) - \nu x'(0)\right).$$

Hence, $|f_{xx}(t, x(t))|$ must be bounded for all $t \in I$. □

Now we are prepared to prove the monotone dependence of an rn-solution $x(t)$ and of its derivative $x'(t)$ on the boundary parameter $S \geq 0$. As a consequence, monotonicity is also obtained for the radial displacement

$$B[x] := x'(1) - \nu\sqrt{S^2 + y^2(1)}\,. \tag{4.4}$$

THEOREM 4.4. *Suppose condition* (Q1) *holds and let* $S_1 > S_2 \geq 0$ *be fixed. Let* x_j, $j = 1, 2$, *denote the respective rn-solutions of the* BVPs (S_j) *and define* $d(t) := x_1(t) - x_2(t)$.

(a) *If condition* $(Q_{\alpha,\beta})$ *holds, then we have* $d(t) > 0, t \in (0, 1]$.
(b) *If the stronger condition* $(Q_{0,\beta})$ *holds, then we have* $d'(t) > 0$, $t \in I$.
(c) *If condition* $(Q_{0,\beta})$ holds with $\beta = 1 - \nu$, *then we have* $B[x_1] > B[x_2]$.

Proof: (a) Let $w > 0$ be defined as in the proof of Theorem 3.5. The function $v := d/w$ solves the differential equation (3.9), which may be written in the form

$$v'' + \left(\frac{2w'(t)}{w(t)} + \frac{1}{t}\right)v' - \frac{h(t)}{t^2}v = 0. \tag{4.5}$$

We have $v(0) = 0$ and $v(1) = (S_1 - S_2)/w(1) > 0$. Since $(Q_{\alpha,\beta})$ holds, the maximum principle applies to (4.5) with the result that $v(t) \geq 0$, $t \in I$. Thus, $v(0) = 0$ is a minimum. Let us prove that $v'(t) > 0$ holds for each $t \in I$. In this case we have $v(t) > 0$ for $0 < t \leq 1$ and the assertion is obtained from $d(t) = v(t)w(t)$.

Clearly, the coefficients $2w'(t)/w(t) + 1/t$ and $-h(t)/t^2$ of equation (4.5) are bounded on every closed subinterval of I_0. Furthermore, in view of Lemma 4.3, the quantity

$$\frac{2w'(t)}{w(t)} + \frac{1}{t} - \frac{h(t)}{t} = \frac{2w'(t)}{w(t)} - \alpha^2 t^{\alpha-1} - f_x(t, x_\mu(t))$$

is bounded at $t = 0$. In this situation, Theorem 4 of [26] applies to $-v$ with the result that $v'(0) > 0$. In constructing a contradiction to the assertion, we shall now assume that there exists a point $t_0 \in (0, 1]$ with $v'(t_0) = 0$ and $v'(t) > 0$ for $0 \leq t < t_0$. From (4.5) and $(Q_{\alpha,\beta})$ if follows that $(tw^2(t)v')' = t^{-1}h(t)w^2(t)v \geq 0$; and therefore

$$0 \leq \int_0^{t_0} (sw^2(s)v'(s))'\, ds = t_0 w^2(t_0)v'(t_0) = 0.$$

Thus, $tw^2(t)v'(t) = \text{const} = 0$ holds for all $t \in [0, t_0]$, which contradicts the assumption. Hence, we have $v'(t) > 0$, $t \in I$.
(b) This assertion follows from (a) by specializing to $\alpha = 0$ and $w = 1$.
(c) For $\nu = 0$, this assertion follows from (b) because of $B[x_1] - B[x_2] = d'(1) > 0$. Utilizing the integral representation (3.6) and the relation (3.7), we obtain in the general case of $\nu > 0$:

$$B[x_1] - B[x_2] > d'(1) - \nu\, d(1) = (1 - \nu)\, d(1) + \int_0^1 D(x_\mu(s))\, d(s)\, ds.$$

In view of the result (b) and condition $(Q_{0,\beta})$ the last integral is bounded below by $-(1 - \nu)\, d(1)$. Thus, $B[x_1] - B[x_2] > 0$ holds, and the lemma is proved. □

5. The displacement problem (H)

Throughout this section we always assume that condition (Q1) is valid, or equivalently that $C > C_0 := \max_{s \in [0,L]} |\nu q_\zeta(s) r(s)/r'(s)|$ holds.

Theorem 3.4 shows that the strip $Ex(S, C) := \{(S, C) | S \geq 0,\ C > C_0\}$ belongs to the domain of existence of rn-solutions of the BVP (S). If in addition condition $(Q_{\alpha,\beta})$ holds, then any point $(S, C) \in Ex(S, C)$ is uniquely related to an rn-solution $x(t)$ and its derivative

$x'(t)$. Denoting these relations by $A_t(S, C) := x(t)$ and $DA_t(S, C) := x'(t)$ respectively, the displacement boundary condition

$$B[x] := DA_1(S, C) - \nu\sqrt{S^2 + y^2(1)} = H \tag{5.1}$$

may be interpreted as a mapping from the set $Ex(S, C)$ onto the set of those parameter values (H, C) which admit for a unique rn-solution of the BVP (H). In further investigations the continuity of A_t and DA_t plays a central role. For a proof of the continuity, we shall proceed in several steps. In a first step, the strong continuity with respect to $S \geq 0$ is established.

LEMMA 5.1. *Suppose condition* $(Q_{0,\beta})$ *holds and let* $C > C_0$ *be fixed. Then* $A_t(S, C)$ *and* $DA_t(S, C)$ *are strongly continuous with respect to* $S \geq 0$.

Proof: For $S_1 > S_2 \geq 0$, we obtain with the notations of Theorem 4.4 $\|A_t(S_1, C) - A_t(S_2, C)\| = \|d\| = d(1) = S_1 - S_2$, and thereby the strong continuity of $A_t(S, C)$. In order to prove strong continuity of $DA_t(S, C)$, we let $x_0(t) := A_t(0, C)$ and observe that $x_0'(0) > 0$. Like in the proof of Lemma 4.3 there exists a constant K_6, depending on $x_0(t)$, such that for all $x(t) \geq x_0(t)$, $t \in I$, we find

$$\left\|\frac{1}{t} f_x(\cdot, x)\right\| \leq \left\|\frac{Cpy^2/t}{(x_0^2 + y^2)^{3/2}}\right\| + \left\|\frac{\nu p^2 qy}{x_0^2 + y^2}\right\| =: K_6 < \infty .$$

Now, with the notation $x_\mu(t) := x_1(t) + \mu(t)(x_2(t) - x_1(t))$, $0 < \mu(t) < 1$, it follows from Theorem 4.4 that $x_\mu(t) \geq x_0(t)$. Using this and the integral representation (3.6), we obtain

$$\|DA_t(S_1, C) - DA_t(S_2, C)\| \leq (S_1 - S_2)\left(1 + \left\|\int_0^1 \left|\partial_t K(t, s) s \frac{f_x(s, x_\mu(s))}{s}\right| \mathrm{d}s\right\|\right) \leq (S_1 - S_2) K_7 ,$$

where $K_7 := (1 + K_6(x_0)/3)$. So the lemma is proved. □

The next step concerning strong continuity with respect to $C > C_0$ is based on a result on implicit functions (Thm. 3.1 in [27], Chapt. X). We take the Banach space $E_0 := \{x \in E \mid \max_{t\in I} |x(t)/t| < \infty\}$, equipped with the norm $\|x\|_0 := \|x/t\|$, and consider the set

$$\Omega := \{x \in E_0 \mid x(t) > 0 , \quad t \in I_0, \quad 0 < \lim_{t \to 0+} \frac{x(t)}{t} < \infty \quad \text{exists}\} .$$

This is an open subset of E_0 related to the topology induced by $\|\cdot\|_0$. Letting $I_C := (C_0, +\infty)$, the set $\Omega \times I_C$ becomes a metric space by virtue of $d(x_1, C_1; x_2, C_2) := \|x_1 - x_2\|_0 + |C_1 - C_2|$. On this space, the nonlinear mapping $N: \Omega \times I_C \to E_0$ will be defined by

$$N(x, C)(t) := x(t) - St + \int_0^1 K(t, s) f(s, x(s)) \,\mathrm{d}s , \quad (x, C) \in \Omega \times I_C , \quad t \in I .$$

Now suppose condition $(Q_{\alpha,\beta})$ holds. Notice that any $C^* \in I_C$ is related to a unique rn-solution x^* of the BVP (S), $S \geq 0$. Since $(x^*)'(0) > 0$, we have $x^* \in \Omega$ and $N(x^*, C^*) = 0$. This justifies the assumption that the equation $N(x, C) = 0$ implicitly defines a function

$x(t) = A_t(S, C)$ which continuously depends on C. In order to see this we prove several claims.

claim 1: The mapping $N: \Omega \times I_C \rightarrow E_0$ is continuous .

Proof: For each $(x, C) \in \Omega \times I_C$ we have $N(x, C) \in E_0$. Indeed, while $N(x, C) = x - Tx$ with T as defined in (3.3), we see that $N(x, C)$ belongs to the space E. Since $\lim_{t \to 0+} x(t)/t > 0$ exists, it further follows from (2.10) that $f(t, x(t))/t \rightarrow -Cp'(0)$ as $t \rightarrow 0+$. So $|f(t, x(t))/t|$ must be bounded by $3(C+1)K_8$ with a constant $K_8(x)$ independent of C but depending on x. Using $t^{-1} \int_0^1 K(t, s)s \, ds = (1-t)/3$, we obtain

$$\left|\frac{1}{t} N(x, C)(t)\right| \leq \left|\frac{x(t)}{t}\right| + S + (C+1)(1-t)K_8 , \quad t \in I ,$$

which shows $N(x, C) \in E_0$. In order to prove the continuity with respect to (x, C), we consider $x, x_1 \in \Omega$ and a fixed $C \in I_C$. We have

$$\|N(x, C) - N(x_1, C)\|_0 \leq \|x - x_1\|_0 \left(1 + \frac{1}{3} \|f_x(\cdot, x_\mu)\|\right) ,$$

with an intermediate variable $x_\mu(t) := x_1(t) + \mu(t)(x(t) - x_1(t))$, $0 < \mu(t) < 1$. Since the function $x_1(t)/t$ is positive on $[0, 1)$, there exists a number $\delta > 0$ such that $\rho(t) := x_1(t)/t - \delta + |y(t)|/t > 0$ holds for all $t \in I$. A restriction of $x \in \Omega$ to the ball $\|x_1 - x\|_0 \leq \delta$ yields $\|x_1 - x_\mu\|_0 \leq \delta$, and in particular $x_\mu(t)/t \geq x_1(t)/t - \delta$. An application of the inequality $(a^2 + b^2)^{n/2} \geq 2^{-n/2}(a+b)^n$ which holds for $a, b \geq 0$ and $n \in \mathbb{N}$, and use of the representation (4.3) furnishes the result

$$\|f_x(\cdot, x_\mu)\| \leq \left\|\frac{\sqrt{8}Ctp(t)}{\rho^3(t)} \left(\frac{y(t)}{t^2}\right)^2\right\| + \left\|\frac{2\nu tp^2(t)q(t)}{\rho^2(t)} \frac{y(t)}{t^2}\right\| \leq 3(C+1)K_9 ,$$

where the constant $K_9(x_1)$ is independent of C. On the other hand, for fixed $x_1 \in \Omega$ and any $C, C_1 \in I_C$, the inequality $\|N(x_1, C) - N(x_1, C_1)\|_0 \leq |C - C_1| K_8(x_1)$ holds. This finally leads to

$$\|N(x, C) - N(x_1, C_1)\|_0 \leq \|x - x_1\|_0 (1 + (C+1)K_9(x_1)) + |C - C_1| K_8(x_1) ,$$

where the triangle inequality has been applied. So we have proved claim 1. □

claim 2: The Fréchet derivative $N_x'(x^*, C^*)$ exists as a bounded linear operator from E_0 to E_0.

Proof: The linearity of the Fréchet derivative becomes obvious from the representation

$$N_x'(x^*, C^*) \, d(t) = d(t) + \int_0^1 K(t, s) H(s) \, d(s) \, ds , \quad d \in E_0 , \quad t \in I ,$$

where $H(t) := f_x(t, x^*(t))$. Now, the boundedness $\|N_x'(x^*, C^*) \, d\|_0 \leq \|d\|_0 (1 + K_4/3)$ follows from Lemma 4.3.

claim 3: For each fixed $(x_1, C_1) \in \Omega \times I_C$, we have

$$\lim_{\substack{I_c \ni C \to C_1 \\ \Omega \ni x \to x_1}} \sup_{\|d\|_0 \leq 1} \|N'_x(x, C)\,\mathrm{d} - N'_x(x_1, C_1)\,d\|_0 = 0\,.$$

Proof: Following mainly the proof outlines of claim 1, we first derive

$$\begin{aligned}\|N'_x(x, C)\,d - N'_x(x_1, C)\,d\|_0 &\leq \|d\|_0\|x - x_1\|_0\|f_{xx}(\cdot, x_\mu)\| \max_{t\in I} t^{-1} \int_0^1 K(t, s)s^2\,\mathrm{d}s \\ &\leq \|d\|_0\|x - x_1\|_0(C + 1)K_{10}\,,\end{aligned}$$

which holds for any $x \in \Omega$ close to $x_1 \in \Omega$ and for a constant $K_{10}(x_1)$ not depending on $C \in I_C$. Likewise, $\|N'_x(x_1, C)\ d - N'_x(x_1, C_1)\,d\|_0 \leq |C - C_1|\ \|d\|_0K_9(x_1)$, and thus the assertion is an immediate consequence of the triangle inequality. □

claim 4: The inverse $F := [N'_x(x^*, C^*)]^{-1}$ exists as a bounded linear operator from E_0 to E_0.

Proof: We use an argument from functional analysis. Letting $K_1(t, s) := K(t, s)H(s)$, an elementary computation shows $K_1 \in L^2(I \times I)$. Thus, in Hilbert space $L^2(I)$ the integral equation

$$d(t) + \int_0^1 K_1(t, s)\,d(s)\,\mathrm{d}s = rs(t)\,, \quad t \in I\,, \tag{5.2}$$

is subordinate to Fredholm's theory. The associated homogeneous equation admits only a trivial solution. In order to see this, we notice that any solution $\mathrm{d} \in L^2(I)$ has regularity $d \in C^2(I_0) \cap C(I)$ and solves the BVP

$$[Ld](t) = (1 + tH(t))\,d(t)\,, \quad t \in I_0\,, \quad d(0) = d(1) = 0\,.$$

While $tH(t) = D(x^*(t))$, this problem was shown in Theorem 3.5 to have the unique solution $d \equiv 0$. Therefore, Fredholm's theory claims that a unique solution $\mathrm{d} \in L^2(I)$ of equation (5.2) exists for each $rs \in L^2(I)$. While $K_1(t, s)$ is continuous, a standard regularity argument shows that any right hand side $rs \in E_0$ generates a solution $\mathrm{d} \in E$ with the property

$$\|d\|_0 \leq \|rs\|_0 + \frac{1}{3}\,\|d\|\ \|f_x(\cdot, x^*)/t\| < \infty\,,$$

and so F exists as a homeomorphism from E_0 to E_0. □

Let us now discuss the continuity of the mappings $A_t(S, C)$ and $DA_t(S, C)$.

LEMMA 5.2. *Suppose condition $(\mathrm{Q}_{\alpha,\beta})$ holds and let $S \geq 0$ be fixed. Then $A_t(S, C)$ and $DA_t(S, C)$ are strongly continuous with respect to $C \in I_C$.*

Proof: After having supplied all prerequisites of the well-known implicit function theorem (cf. [27], Thm. 3.1, Chapt. X), it follows from claim 1 to claim 4 that $A_t(S, C)$, measured in $\|\cdot\|_0$-norm, is continuous with respect to $C \in I_C$. Since $\|x\| \leq \|x\|_0, x \in \Omega$, continuity holds also with respect to the weaker topology induced by the $\|\cdot\|$-norm. As for the continuity of $DA_t(S, C)$, we notice that an elementary computation yields

$$\|DA_t(S, C_1) - DA_t(S, C_2)\| \leq |C_1 - C_2| K_8(x_1) + (C_2 + 1)\|x_1 - x_2\|_0 K_9(x_1),$$

where $x_j(t) := A_t(S, C_j)$, $j = 1, 2$, and where the constants K_8, K_9 have previously been introduced, observing the restriction $\|x_1 - x_2\|_0 \leq \delta$. Now, continuity as claimed is immediately obtained from the continuity of A_t in $\|\cdot\|_0$-norm. □

Combining the results of Lemma 5.1 and Lemma 5.2, we finally have:

LEMMA 5.3. *Suppose condition* $(Q_{0,\beta})$ *holds. Then the mappings* $A_t(S, C)$ *and* $DA_t(S, C)$ *are strongly continuous at each point* $S \geq 0$, $C > C_0$.

Since strong continuity implies weak continuity at each point $t \in I$, the displacement boundary condition (5.1) induces a mapping

$$\gamma_C(S) := B[x] = DA_1(S, C) - \nu\sqrt{S^2 + y^2(1)}, \quad S \geq 0, \tag{5.3}$$

which continuously depends on $S \in \overline{\mathbb{R}}_+ := [0, \infty)$ and $C \in I_C$. The following properties are of importance.

LEMMA 5.4. *Suppose condition* $(Q_{0,\beta})$ *holds with* $\beta = 1 - \nu$. *Then*:
(a) *For any fixed* $C > C_0$, *the mapping* $\gamma_C : \overline{\mathbb{R}}_+ \to \mathbb{R}$ *is strictly increasing and continuous. The range of* γ_C *is the segment* $[\gamma_0(C), +\infty)$, *where* $\gamma_0(C) := \gamma_C(0)$.
(b) *The mapping* $\gamma_0 : I_C \to \mathbb{R}$ *is continuous and admits the representation* $\gamma_0(C) = DA_1(0, C) - \nu|y(1)|$.

Proof: In view of Lemma 5.3 and Theorem 4.4 nothing is needed except for $\gamma_C(S) \to +\infty$ as $S \to +\infty$. This, however, is easily seen using

$$B[x] \geq x'(1) - \nu(S + |y(1)|) = (1 - \nu)S + \int_0^1 sf(s, x(s))\,ds - \nu|y(1)|,$$

with $1 - \nu > 0$ and with the uniform bound $|f(t, x(t))| \leq K_1$. □

The idea is now to identify the set $Ex(H, C) := \{(H, C) | H \geq \gamma_0(C),\ C > C_0\}$ as the homeomorphic image of the set $Ex(S, C)$, and therefore as the only set of boundary parameters where a unique rn-solution of the BVP (H) exists. Indeed, any fixed $(S, C) \in Ex(S, C)$ is related to an rn-solution $x(t)$ to which a radial displacement $H := \gamma_C(S) \geq \gamma_0(C)$ corresponds. Thus, the set $Ex(H, C)$ must enclose all boundary data of possible rn-solutions as far as $C > C_0$ holds. Conversely, given a fixed $(H, C) \in Ex(H, C)$, then H belongs to the range of γ_C. Since this mapping is one-to-one, a unique $S \geq 0$ exists with $H = \gamma_C(S)$. Therefore, the rn-solution $x(t)$ related to the point $(S, C) \in Ex(S, C)$ induces the displacement H.

A displacement $H < \gamma_0(C)$ for instance does not belong to the range of γ_C, and hence cannot be induced by any rn-solution. Finally, to any displacement $H = \gamma_0(C)$ there corresponds an rn-solution $x(t)$ which satisfies $x(1) = 0$. Thus, the set $Ex(H, C)$ has a continuous boundary manifold

$$\Gamma := \{(H, C) \mid H = \gamma_0(C), C > C_0\}\ , \tag{5.4}$$

that separates the domains of existence and non-existence of tensile solutions of the BVP (H). Let us summarize what we have proved up to this point.

THEOREM 5.5. *Suppose conditions* (Q1) *and* $(\mathrm{Q}_{0,\beta})$ *hold with* $\beta = 1 - \nu$. Let $x_0(t) := A_t(0, C)$ *denote the unique rn-solution of the* BVP (S), $S = 0$, *and define* $\gamma_0(C) := x_0'(1) - \nu|y(1)|$, $C \in I_C$. *Then the continuous curve* Γ *decomposes the parameter range* $(H, C) \in \mathbb{R} \times I_C$ *into complementary subsets of existence and non-existence as follows*:

(a) *For each* $C \in I_C$ *there exists a unique rt-solution* $x(t)$ *of the* BVP (H) *if and only if* $H > \gamma_0(C)$.
(b) *For each* $C \in I_C$ *and* $H = \gamma_0(C)$ *there exists a unique rn-solution* $x(t)$ *of the* BVP (H) *which coincides with the function* $x_0(t; C)$ *above and thus satisfies* $x(1) = 0$.

6. Wrinkle-free solutions

In the preceding sections mainly the problem of determining the set of those parameters S, C was discussed which are related to a nonnegative radial stress resultant $M(s) = Lx(t)$. Numerical examples show that a membrane state exists with a positive radial stress $x(t)$ but a circumferential stress resultant S_θ which is negative in parts of the membrane. In view of (2.1), (2.2a), (2.3) and (2.5), we have

$$S_\theta(s) = \frac{\mathrm{d}}{\mathrm{d}s} M(s) = (1 + sr_1(s)) \exp\left(-\int_s^L r_1(\tau)\, \mathrm{d}\tau\right) x'(t)\ , \quad t \in I\ . \tag{6.1}$$

Thus, an rn-solution $x(t)$ is wrinkle-free if and only if $x'(t) \geq 0$ holds for all $t \in I$. Notice that the rn-solution $x(t)$ of the BVP (S), $S = 0$, is not wrinkle-free since $x'(t_0) < 0$ holds at least for one point $t_0 \in (0, 1]$. Wrinkle-free solutions are just the monotone increasing rt-solutions.

In order to determine the subset $W(S, C) \subset Ex(S, C)$ of those parameters $S > 0$ and $C > C_0$ which belong to a wrinkle-free solution of the BVP (S), it is convenient to introduce a mapping

$$g(S, C) := \min_{t \in I} DA_t(S, C)\ , \quad (S, C) \in \mathbb{R}_+ \times I_C\ , \tag{6.2}$$

which measures the monotonicity of an rn-solution $x(t) = A_t(S, C)$ dependent on S and C. The following result on the existence of an implicitly defined Γ_S is vital.

LEMMA 6.1. *Suppose conditions* (Q1) *and* $(\mathrm{Q}_{0,\beta})$ *hold. Then there exists a uniquely defined continuous function* $\Gamma_S : I_C \to \mathbb{R}$, $\Gamma_S(C) = S > 0$ *satisfying* $g(\Gamma_S(C), C) = 0$ *for all* $C \in I_C$.

Proof: Lemma 5.3 shows the continuity of $g(S, C)$ at each point $(S, C) \in \mathbb{R}_+ \times I_C$. Moreover, $g(S, C)$ as a function of $S > 0$ is strictly increasing. For a proof, let $C > C_0$ be fixed and consider $S_1 > S_2 > 0$. Using Theorem 4.4(b), we obtain

$$\begin{aligned} g(S_1, C) - g(S_2, C) &= \min_{t \in I} DA_t(S_1, C) - \min_{t \in I} DA_t(S_2, C) \\ &\geq \min_{t \in I} (DA_t(S_1, C) - DA_t(S_2, C)) > 0\ . \end{aligned}$$

As mentioned above, we have $g(0, C) < 0$, and from the integral representation (3.6) it follows that $x'(t) > 0$ is equivalent to

$$S > \frac{1}{2} \int_0^1 (1 - s^2) \frac{1}{s} f(s, x(s))\, ds - \frac{1}{2}\left(1 + \frac{1}{t^2}\right) \int_0^t sf(s, x(s))\, ds\,, \quad t \in I\,. \tag{6.3}$$

Using $-Cp(0) \leq f(t, x(t)) \leq -Ctp'(t) + \nu tp^2(t)q(t)$, condition (6.3) holds at least for all

$$S \geq S^* := \frac{1}{2} \int_0^1 (-Cp'(s) + \nu p^2(s)q(s))\, ds + \frac{1}{2} Cp(0) > 0\,.$$

So we have $g(S^*, C) > 0$, and due to the monotonicity, for each $C > C_0$, there exists a unique $S > 0$ with $g(S, C) = 0$. It follows from a theorem on implicit functions (cf. [28], Satz 51), that the induced mapping $\Gamma_S(C) := S$ has the requested properties. □

The curve $\Gamma_S(C)$ characterizes the subdomain $W(S, C)$ of wrinkle-free solutions as follows.

THEOREM 6.2. *Suppose conditions* (Q1) *and* ($Q_{0,\beta}$) *hold. Then the domain of existence and uniqueness of wrinkle-free solutions of the* BVP (S) is the set

$$W(S, C) := \{(S, C) \mid S \geq \Gamma_S(C)\,, \quad C > C_0\}\,. \tag{6.4}$$

For fixed $C > C_0$ and $S > \Gamma_S(C)$, the circumferential stress S_θ is strictly positive everywhere in the membrane while for $S = \Gamma_S(C)$ there exists one point where S_θ vanishes.

Proof: The set $W(S, C)$ is evidently a subset of $Ex(S, C)$. Concerning the positivity of S_θ, all assertions are immediate consequences of the monotonicity of $g(S, C)$. □

In order to determine the domain of wrinkle-free solutions of the BVP (H), the separatrix $\Gamma_S(C)$ is mapped into the parameter domain of the displacement data, using (5.3). It follows from Lemma 5.4 and Lemma 6.1 that the result

$$\Gamma_H(C) := DA_1(\Gamma_S(C), C) - \nu\sqrt{\Gamma_S^2(C) + y^2(1)}\,, \quad C > C_0\,, \tag{6.5}$$

is a continuous curve which characterizes the subset $W(H, C) \subset Ex(H, C)$ of wrinkle-free solutions as follows.

THEOREM 6.3. *Suppose conditions* (Q1) *and* ($Q_{0,\beta}$) *hold with $\beta = 1 - \nu$. Then the domain of existence and uniqueness of wrinkle-free solutions of the* BVP (H) *is the set*

$$W(H, C) := \{(H, C) \mid H \geq \Gamma_H(C), C > C_0\}\,. \tag{6.6}$$

For fixed $C > C_0$ and $H > \Gamma_H(C)$, the circumferential stress S_θ is strictly positive everywhere in the membrane while for $H = \Gamma_H(C)$ there exists at least one point where S_θ vanishes.

The special case of a *flat* circular membrane has been studied in [14] where the analysis was governed by the nonlinear membrane equations of the simplified Reissner model. In this case a maximum principle holds for the circumferential stress showing that the minimum of $S_\theta(s)$ is attained at the edge $s = L$. With this additional information it has been possible to

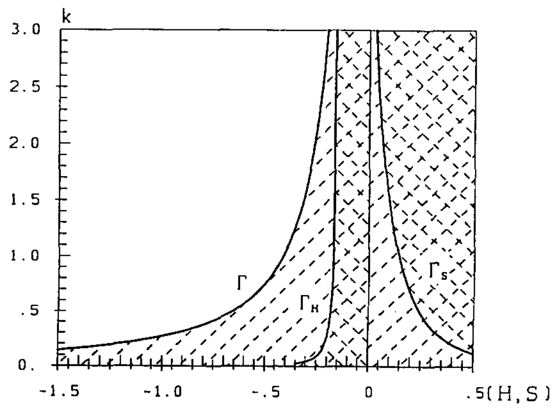

Fig. 1. The arc Γ denotes the separatrix between the domains of existence (dashed) and non-existence (blank) of tensile solutions to the BVP (H). The domains of wrinkle-free solutions (crosshatched) which extend unboundedly to the right are bounded to the left by the arcs Γ_S for Problem (S) and Γ_H for Problem (H). The membrane is *flat*, the load uniform and Poisson's ratio is $\nu = 0.33$.

discuss analytic properties of the separatrices Γ_S and Γ_H such as monotonicity and an asymptotic behavior for $C \to 0+$. In the present approach, an analytic determination of the point $t \in I$, where the minimum of the function (6.2) is attained, seems to be impossible. However, a computational calculation of Γ_S and Γ_H as well as of the curve Γ is a standard problem of solving some boundary value problems numerically. For example, a calculation of Γ requires the numerical integration of the boundary value problem

$$Lx = x + F(t, x)\ , \quad t \in I_0\ , \quad x(0) = 0 = x(1)\ , \quad C > C_0\ .$$

Having determined the numerical solution $\tilde{x}_0(t) = \tilde{A}_t(0, C)$ and its first derivative, an approximation of Γ is obtained, letting in (5.4) $\gamma_0(C) \doteq \tilde{x}'(1) - \nu|y(1)|$. Some computational results are documented in Figs. 1 and 2 for a flat circular membrane $r(s) := s$ (i.e. $p(t) \equiv 1$, Fig. 1) and for a curved membrane $r(s) := (4L/\pi)\sin(\pi s/4L)$ (i.e. $p(t) = ((8/\pi)\tan(\pi/8))/(1 + t^2\tan^2(\pi/8))$, Fig. 2), assuming $\nu := 0.33$ and a uniform surface load $q = 1$. The notation $k := 1/C$ is used.

Calculating the quantity $k_0 := \min_{s \in [0,L]}(Lr'(s)/\nu r(s))$, it can be seen immediately that

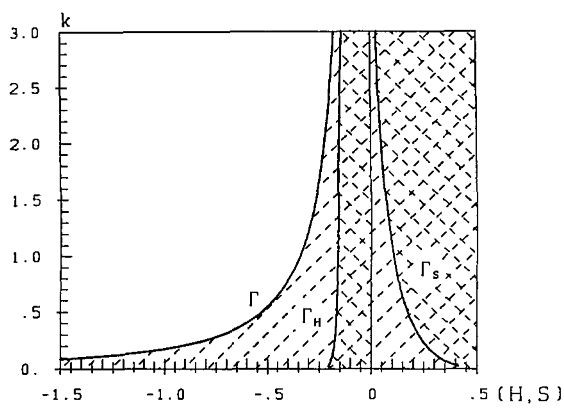

Fig. 2. Domains of existence and non-existence of tensile and wrinkle-free solutions for a *curved* membrane under a uniform load and for Poisson's ratio $\nu = 0.33$.

condition (Q1) holds within the range $0 < k < k_0 \doteq 3.03$ (flat membrane) and within $0 < k < k_0 \doteq 2.38$ (curved membrane), while condition $(Q_{\alpha,\beta})$ can be replaced by $(Q''_{\alpha,\beta})$ which holds for $\alpha = 0$ and $\beta = 1 - \nu \geqslant g(1)$ as remarked previously. Numerical computations show that positive solutions do exist, even for $2.38 \leqslant k \leqslant 3$, although (Q1) is violated in case of the curved membrane.

7. Concluding remarks

We have considered rotationally symmetric deformations of a curved circular membrane under the action of a variable vertical surface load. An additional edge force occurred from prescribing either a constant radial stress S or a constant radial displacement H at the membrane edge. The interior radial stress which arises from deformation was expressed in terms of the solutions of a nonlinear second order ODE. We have discussed two different kinds of boundary value problems according to the different types of edge loads, namely the stress problem (S) and the displacement problem (H). The boundary data S and H together with a system parameter $C = Eh$ (dependent on the thickness of the membrane and the modulus of elasticity) constitute the parameter varieties of these problems. In a first step, using an integral equation technique and a maximum principle, we have shown that under reasonable assumptions on the surface load at each point of the parameter strip $S \geqslant 0$, $C > C_0$ there exists a unique tensile solution of the stress problem (Thm. 3.4 and Thm. 3.5). The result is best possible in the sense that the lower bound C_0 which depends on the shape function of the membrane, on Poisson's ratio and on the magnitude of the surface load cannot be improved.

Physically, a tensile solution represents a meridional stress component which is nonnegative everywhere within the membrane. In order to prevent wrinkling of the stretched membrane the circumferential stress component S_θ must also be nonnegative everywhere. Tensile solutions which are associated with a principal stress $S_\theta \geqslant 0$ have been termed wrinkle-free. Thus, the condition for incipient wrinkling of a tensile solution is the vanishing of S_θ at some point of the membrane. Accordingly, we have discussed analytically and numerically the set of those parameters S, C, for which S_θ has a zero. This set has been shown (Thm. 6.2) to form a continuous curve $S = \Gamma_S(C)$ in (S, C)-parameter space separating wrinkle-free solutions from unstable solutions.

Analogous results on existence and uniqueness of tensile and wrinkle-free solutions have also been derived for the displacement problem by employing a mapping argument. We have used monotonicity and continuity properties (Thm. 4.4, Lemma 5.3) of the displacement boundary operator in order to map the straight line $S = 0, C > C_0$ onto a continuous curve $H = \gamma_0(C)$. This curve has been shown to subdivide the parameter variety H, C of the displacement problem into complementary sets of existence and nonexistence of tensile solutions (Thm. 5.5). A similar subdivision separating wrinkle-free solutions from unstable solutions has finally been obtained by the image $H = \Gamma_H(C)$ of the curve Γ_S (Thm. 6.3).

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